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# Fuzzy logics as the logics of chains

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**Abstract**

The paper proposes a formal delimitation of the class of 'fuzzy logics' and answers some objections that can be raised against the definition; the focus is put on informal and motivational aspects of the problem.


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*Keywords:* Non-classical logics; Formal fuzzy logic; Weakly implicative fuzzy logics

## 1. Introduction

There is a family of logics that have certain common features and applications and that are usually called 'fuzzy'. Although their investigation advanced rapidly during the past years, there is as yet no formal delimitation of this class of logics. This paper proposes a formal definition of the class of 'fuzzy logics', and answers some objections that can be raised against the definition. We focus on the informal aspects of the problem; technical details are described in [1,2]. We assume some familiarity of the reader with usual fuzzy logics investigated in the literature.

## 2. Weakly implicative logics

Since the notion of logical system is so wide and flexible that it can hardly be captured by a formal definition, we do not attempt to cover all possible kinds of formalism in which fuzzy logics can be rendered. Instead, we delimit the class of fuzzy logics within some basic class of well-defined logics, which nevertheless is broad enough to contain many usual systems of logics.

Although there are many systems of first-order (or even higher-order) fuzzy logics, the basic difference between fuzzy logics and other kinds of logics lies in the propositional level, namely in the interpretation of propositional connectives. Therefore, we first restrict our attention to propositional logics. By 'propositional logic' we mean a logic with the classical style of syntax (i.e., defined inductively from propositional variables and $n$-ary connectives) and

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given by a *consequence relation* between *sets of formulae* (premises) and *formulae* (conclusion), closed under arbitrary substitutions.

We impose some further (rather weak) restrictions on the consequence relation, thus arriving at the class of *weakly implicative logics* (introduced by Cintula in [2]); it extends Rasiowa's well-known class of *implicative logics* of [9] by omitting the rule of weakening. This class, although rather narrow from the point of view of Abstract Algebraic Logic (see [4]), is broad enough to contain many 'usual' logics, while still possessing good logical properties.

**Definition 1** (*Weakly implicative logics*). A logic (represented by the consequence relation $\vdash$ closed under substitutions) is *weakly implicative* iff it contains a (definable) connective $\to$ that satisfies the following conditions:

$$\vdash \varphi \to \varphi, \tag{1}$$

$$\varphi,\ \varphi \to \psi \vdash \psi, \tag{2}$$

$$\varphi \to \psi,\ \psi \to \chi \vdash \varphi \to \chi, \tag{3}$$

$$\varphi \to \psi,\ \psi \to \varphi \vdash c(\ldots, \varphi, \ldots) \to c(\ldots, \psi, \ldots), \ \textit{for all connectives } c. \tag{4}$$

**Example.** The following classes of logics are weakly implicative: superintuitionistic logics, classical modal logics, substructural logics (rendered as Hilbert-style calculi), all usual fuzzy logics with classical syntax, all of Rasiowa's implicative logics, etc. Logics with non-classical syntax (e.g., labelled deduction, evaluated syntax, etc.) as well as logics with connectives that are not congruent w.r.t. $\leftrightarrow$ (e.g., subclassical modal logics) fall outside the class.

Weakly implicative logics can be characterized as those which are complete w.r.t. a class of (*pre*)*ordered matrices* (in which the set $D$ of designated values is upper), if the ordering of the elements of the matrix $\mathbf{M}$ is defined as

$$x \leqslant_{\mathbf{M}} y \ \equiv_{\mathrm{df}}\ x \to_{\mathbf{M}} y \in D. \tag{5}$$

In more words, recall the following definition of matrix semantics and the strong completeness theorem that apply to weakly implicative logics (see [2]):

**Definition 2** (*Matrix semantics*). A *matrix* $\mathbf{M} = (\mathbf{A}, D)$ for a language $\mathcal{L}$ is an $\mathcal{L}$-algebra $\mathbf{A}$ plus its subset $D$ of *designated values.* The interpretation of a connective $c$ of the language $\mathcal{L}$ in an $\mathcal{L}$-matrix $\mathbf{M}$ is denoted by $c_{\mathbf{M}}$. We say that an evaluation $e$ in $\mathbf{M}$ *validates* a formula $\varphi$ iff $e(\varphi) \in D$. The relation of *semantic consequence* $X \models_{\mathbf{M}} \varphi$ holds iff each evaluation in $\mathbf{M}$ that validates all formulae in $X$ validates $\varphi$ as well. Finally, $\mathbf{M}$ is a matrix for the logic $\mathbf{L}$ represented by the consequence relation $\vdash_{\mathbf{L}}$ (an $\mathbf{L}$-*matrix*) iff $X \vdash_{\mathbf{L}} \varphi$ implies $X \models_{\mathbf{M}} \varphi$ (for all $X$ and $\varphi$).

**Theorem 3** (*Strong completeness*). *Let* $\mathbf{L}$ *be a logic represented by the consequence relation* $\vdash_{\mathbf{L}}$, $\varphi$ *be a formula and* $X$ *a set of formulae* (*all in the language of* $\mathbf{L}$). *Then* $X \vdash_{\mathbf{L}} \varphi$ *iff for all* $\mathbf{L}$-*matrices* $\mathbf{M}$, $X \models_{\mathbf{M}} \varphi$.

Notice that the requirements (1) and (3) ensure that the relation $\leqslant_{\mathbf{M}}$ defined on an $\mathbf{L}$-matrix $\mathbf{M}$ by (5) is reflexive and transitive (so it is a preorder). Since we further require the congruence (4) of all connectives w.r.t. bi-implication, w.l.o.g. $\mathbf{M}$ can be factorized to a *partially ordered* matrix (by identifying all elements $x, y$ such that both $x \to_{\mathbf{M}} y \in D$ and $y \to_{\mathbf{M}} x \in D$, i.e., $x \leqslant_{\mathbf{M}} y$ and $y \leqslant_{\mathbf{M}} x$). Finally, the internalization of modus ponens (2), which corresponds to the reasonable assumption that truths imply only truths, requires that the set of designated values $D$ be upper in $\leqslant_{\mathbf{M}}$.

## 3. Weakly implicative *fuzzy* logics

In the previous section we have seen that a logic $\mathbf{L}$ is weakly implicative iff $\mathbf{L}$ is strongly complete w.r.t. the class of all $\mathbf{L}$-matrices which are partially ordered by $\to$. Weakly implicative logics thus can be seen as logics of *partially ordered* matrices. From these logics we select those that by our opinion deserve the name 'fuzzy'.

Our thesis is that *fuzzy logics* are those which are complete w.r.t. *totally ordered* matrices. The thesis is embodied in the following stipulative definition:

**Definition 4** (*Weakly implicative fuzzy logic*). A weakly implicative logic **L** is to be called *fuzzy* iff it is strongly complete w.r.t. the class of all linearly ordered **L**-matrices.

It turns out that this class approximates well the bunch of logics studied in so-called 'fuzzy logic in narrow sense'. In order to avoid confusion and for the sake of simplicity, in what follows we write 'fuzzy logic' in a pretheoretical sense and '*fuzzy* logic' in the technical sense of weakly implicative fuzzy logic. Below we give some reasons why the class could possibly be considered as the definition of the agenda of mathematical fuzzy logic. First, we illustrate the point by some examples.

**Examples.** The following logics, which are usually considered fuzzy, fall indeed within the class of weakly implicative fuzzy logics:

- MTL, BL, and their schematic extensions (IMTL, G, Ł, Π, …).
- Basic hoop logic and its extensions.
- Expressively rich fuzzy logics (ŁΠ, PŁ, …).
- Superintuitionistic logics stronger than G, etc.

On the other hand, the following logics which are usually not considered fuzzy, fall indeed outside the class:

- Intuitionistic logic, BCK.
- Usual modal logics (K, S4, S5, …).
- Full Lambek calculus, linear logic.
- Intermediary logics not stronger than G, etc.

It thus can be seen that the definition at least roughly approximates the usual practice of the fuzzy community. Furthermore, it can be argued (see e.g., [5]) that fuzzy logic investigates the *comparative notion of truth* or *degrees* of truth. Then it is natural to assume that the truth-values, if they are to represent *degrees,* are *comparable*, i.e., totally ordered. The reason why the elements of, say, S4-algebra are usually not construed as the *degrees* of truth—and, consequently, why modal logics are usually not considered fuzzy—seems to consist in the fact that they do not form a scale, i.e., a total order (nor can they be decomposed into totally ordered factors).

Another reason for the suitability of the definition is metamathematical (more exactly, meta-meta-mathematical): it consists in the observation that several methods commonly used in metamathematics of fuzzy logic work *exactly* for (finitary) logics complete w.r.t. linearly ordered matrices. Namely, the subdirect representation, proofs by cases on the reverse implications, and the construction of linear theories for the proof of the completeness theorem. This is based on the following theorem of [2]:

**Theorem 5.** *Let* **L** *be a finitary logic* (*i.e., a logic defined by finitary rules only*). *Then the following are equivalent*:

- (The definition of weakly implicative fuzzy logics.) **L** *is complete w.r.t. the class of all linearly ordered* **L**-*matrices*.
- (Subdirect representation property.) *Each* **L**-*matrix is a subdirect product of linear ones*.
- (Linear extension property). *Each theory in* **L** *can be extended to one whose Lindenbaum-Tarski matrix is linear*.
- (Prelinearity property). $T,\ \varphi \to \psi \vdash \chi$ *and* $T,\ \psi \to \varphi \vdash \chi$ *entails* $T \vdash \chi$ (*for any theory* $T$ *and formulae* $\varphi, \psi, \chi$).

Thus, independently of our proposal to take the definition for the delimitation of the agenda of mathematical fuzzy logic, it is certainly an interesting class of logics which corresponds to properties that play an important role in the metamathematics of fuzzy logic, and is therefore worth studying.

Our definition of *fuzzy* logic is restricted to the class of weakly implicative logics and does not decide which logics outside the class are to be considered fuzzy and which not fuzzy. In particular, the definition does not tell anything about logics which employ non-classical syntax. However, if the logic can be interpreted in a weakly implicative logic, the criterion works well: e.g., the logic with evaluated syntax of [8] comes out fuzzy (as it can be interpreted in Łukasiewicz logic with truth-constants, which is *fuzzy*), while e.g., most substructural logics (given as sets of sequents) do not (except for some which can be argued to be both substructural and fuzzy).

## 4. Answers to possible objections

In this section we answer various objections that have been (or could have been) raised against our thesis. [3] We start with general objections against the very idea of formal definition of fuzzy logics, and proceed with complaints that our definition is too wide or too narrow (we have met both).

**Objection.** *The class of fuzzy logics cannot be characterized mathematically. Only the practice can show which logics are to be called 'fuzzy'.*

**Answer.** It is true that informal notions, being intuitive and often quite vague, cannot be completely captured in a formal system. Nevertheless, it is meaningful to seek mathematical correlates of intuitive notions, as it can yield new insights (witness Church's thesis!), while informal pragmatic definitions are useless for the theory. Although we cannot claim that our definition renders exactly the intuitive notion of fuzzy logic (which is rather vague and probably differs between any two people), it nevertheless approximates well the practice of the community of mathematical fuzzy logic (so it does follow what practice shows us), and can further help precisify the informal notion.

**Objection.** *There is no need to define which logics are fuzzy. For instance, there is no formal definition which logics are modal, and the area of modal logics can do without it.*

**Answer.** Nevertheless, a formal delimitation of the scope of mathematical fuzzy logic can have a unifying influence on the area: the possibility to quantify over all fuzzy logics enables very general formulations of some theorems and can entice the community to seek general methods applicable to wide classes of logics. Furthermore, a formal definition clarifies the position of fuzzy logics in the logical landscape, highlights its relations to neighbouring areas (e.g., substructural logics), and may remind the community of related (but neglected) logics for which fuzzy methods can be adapted.

**Objection.** *The class of logics is interesting, however, it should be given a more neutral name.*

**Answer.** The term 'fuzzy logic' has not as yet been assigned a meaning in *formal* logic. The arguments presented here convince us that the class reconstructs the usual usage of 'fuzzy logic' well, and is of sufficient metamathematical importance to justify usurping the name. Recent developments in fuzzy logic seem to further support our claim, as the remaining parts of our class start to be studied by the community (cf. the uninorm logics of [7], which are *fuzzy* logics without weakening). Thus, it seems that a new term would be superfluous. (Yet if anybody cannot accept the simple name, we have no objections to calling the class e.g. 'fuzzy logics in Cintula's sense'.)

**Objection.** *Traditionally, a logic is fuzzy if its set of truth-values is* [0, 1]. *Fuzzy logic should be connected with real numbers.*

**Answer.** The current practice has already emancipated from [0, 1] to a large degree, as the general semantics of fuzzy logics uses various classes of residuated lattices. Historically, [0, 1] played a role in the development of fuzzy logic (and in applications it still does), but historical reasons are not sufficient for sticking to an unconvenient definition. From the point of view of formal logic, [0, 1] is a complicated (second-order) notion, while linear ordering is a simple first-order concept. Abstracting from the particular model [0, 1] to arbitrary linear orderings seems to us the obvious step, since the most important metamathematical methods used in fuzzy logic (cf. Theorem 5) do not depend on

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[0, 1], but only on the linear ordering of truth values. Furthermore, it is not clear what 'be connected with [0, 1]' should mean:

- The original motivation of a logic?—This would be but another useless informal pragmatic criterion.
- Completeness w.r.t. a [0, 1]-based semantics?—However, by this criterion many predicate fuzzy logics (e.g., Łukasiewicz) would not come out fuzzy (predicate fuzzy logics are generally not standard complete).
- That algebras on [0,1] generate the corresponding variety?—This seems to be too complex a condition. The standard completeness of many logics that are clearly fuzzy (BL, ΠMTL) is either not yet known or was proved only long after the logic had been introduced; on the other hand, linear subdirect decomposability is usually much easier (often it is one of the first results on a new logic).

Thus, we are convinced that the important feature of the semantics of fuzzy logics is not the interval [0, 1], but only its linear order. (Of course if necessary, particular subclasses of *fuzzy* logics which are in some sense related to [0, 1] can be defined and studied—e.g., *core fuzzy* logics of [1].)

**Objection.** *The class includes classical logic, which is not fuzzy.*

**Answer.** We think first that a good definition should include its limit cases (the class contains, for that matter, also the 'contradictory' one-valued logic). If necessary, they can be excluded in particular theorems; but to exclude them in the general definition would be inconvenient, since most results on (non-classical) fuzzy logics hold for Boolean logic as well. Furthermore, it can be argued that classical logic *is* a (very special) representative of fuzzy logics, weird though it may seem: fuzzy logic is a *generalization* of classical logic, designed to encompass *also* vague predicates. Thus, classical logic is a fuzzy logic usable in those special occasions when *by chance* the realizations of all predicates are crisp.

**Objection.** *Fuzzy logics must be distinguished from finitely valued logics. Fuzziness always involves* continuous *change; the matrices should also be dense.*

**Answer.** The previous answer applies: finitely valued logics are fuzzy logics for the special occasions when by chance all propositions in the language acquire only finitely many values (e.g., the multiples of 0.1); and to exclude them from the definition would be inconvenient, as most theorems hold for finite many valued logics as well. As regards the requirement of density, notice that sufficiently large finite number of values effects 'continuous change' as well (cf. the $2^{16}$ colours on computer screens). Nevertheless, such more fine-grained distinctions *within* the class of *fuzzy* logics are welcome.

**Objection.** *Fuzzy logics should have the rule of exchange. There are no natural real-life examples involving non-commutative conjunction* (*except perhaps temporal, but such phenomena should be handled by temporal rather than fuzzy logics*).

**Answer.** Maybe, but most methods of metamathematics of fuzzy logic work with non-commutative linear matrices as well, so it is better to be as general as possible. The extra assumption (of commutativity) would require an extra explanation why it should be imposed (the proof burden is on the proponent of extra assumptions); the lack of examples seems to be a weak reason for this restriction since such examples can be found in the future. Also it is far from obvious that temporal conjunction cannot be modelled just by a non-commutative conjunction instead of a temporal modality.

**Objection.** *Fuzzy logics should have weakening* (*so only* 1 *be designated in the matrices*). *There can be no 'truer truths' than the 'full truth'* 1.

**Answer.** The previous answer applies: the methods work without weakening, and demanding the presence of weakening would require a justification. Moreover, in some circumstances it can be reasonable to assume a *hierarchy* of 'full truths': for instance, men over 7 feet are definitely tall (it is a full truth that they are tall), but still some of them are taller than others.

**Objection.** *The class contains some logics which are usually not considered fuzzy* (*e.g.*, *the relevance logic with mingle* RM).

**Answer.** We are convinced that such logics *should* be regarded and studied as fuzzy, too, as they have a natural interpretation of the degrees of truth. We can say that they are *both* relevant *and* fuzzy—in a similar way as Gödel logic is both superintuitionistic and fuzzy. This is an example of how the formal definition can make us aware that some logic can be studied by the methods of fuzzy logic.

**Objection.** *Some logics*, *e.g.*, psBL, *are studied by fuzzy logicians and usually considered fuzzy*, *though they lack completeness w.r.t. linear matrices*.

**Answer.** Such logics *should not* be considered fuzzy (we stipulate), as they are not the logics of the *comparative notion of truth*. We think that their similarity to fuzzy logics is superficial and that accepting them as fuzzy was not based upon good criteria (analogically, intuitionistic logic also resembles G, but is not fuzzy). Our stipulation also reflects the common discontent of many fuzzy logicians with the absence of the linear subdirect decomposability in these logics, which results in the search for additional axioms that would ensure decomposability (e.g., Kühr's axioms for psBL and psMTL in [6]). Thus, in our opinion, only the decomposable ('representable') extensions $\mathrm{psBL}^r$ and $\mathrm{psMTL}^r$ of psBL and psMTL are to be regarded as fuzzy logics, not psBL or psMTL themselves.

**Objection.** *Some graded notions have incomparable degrees*, *because they have more components* (*e.g.*, *human intelligence*).

**Answer.** It should be stressed that linear matrices are not the only matrices for *fuzzy* logics; the definition only requires that the matrices are subdirectly decomposable into linear ones (Theorem 5). Thus, a particular algebra for a real-life application can always be non-linear; it will have a subdirect representation, though, whose projections correspond to the components of the complex graded notion. These components are usually construed as linear (comparable), or can at least be believed to be further decomposable into linear quantities. For example, in the case of human intelligence, we suppose that it has such components as memory, visualization abilities, verbal skills, etc. These can perhaps be measured on linear scales, or we may believe that they are decomposable into some more elementary ones, which finally could (this is just a conceptual model, but a very natural one); and if, on the contrary, we believe that no such decomposition is possible, we also cast doubt at the original assumption that intelligence is a *graded* notion to which fuzzy logic is applicable. The objection in fact supports our thesis, as it is in accordance with the usual conception of complex graded notions.

The list of objections and answers is of course far from complete and conclusive; we welcome any new objections as well as counter-arguments to our answers, so that the discussion can continue.

## 5. Further prospects

There are numerous results about relations among particular *fuzzy* logics (whether they are stronger or weaker, conservative extensions, fragments, etc.). There are also results relating *fuzzy* logic with some non-*fuzzy* ones. However, we believe that only the formal definition of the notion of fuzzy logic can lead to the systematic development of this class. We give here just a few hints of the possible directions in this research.

The following theorem proved in [2] together with the fact that the class of *fuzzy* logics is defined formally allows us to speak about the weakest *fuzzy* logic with some properties:

**Theorem 6.** *The intersection of any system of* fuzzy *logics is a* fuzzy *logic*.

This allows us (see [2]) to extend the known result of [3] that MTL is an extension of $\mathrm{FL_{ew}}$, to the following one: MTL is the *weakest fuzzy* logic that extends $\mathrm{FL_{ew}}$. In fact, for each logic we can assign the weakest *fuzzy* logic extending it. This turns out to be an important methodological guideline: the weakest *fuzzy* logic possessing some property seems to be a good starting point for the investigation of the property in the context of fuzzy logic. We present some examples:

**Example.** Gödel logic is the weakest *fuzzy* extension of intuitionistic logic. Similarly, $\mathrm{psMTL}^r$ is the weakest *fuzzy* extension of $\mathrm{FL_w}$ [6], MTL of $\mathrm{FL_{ew}}$ [3], and IMTL of $\mathrm{AMALL}^-$ [3].

Another example of a general notion definable by a reference to the class of *fuzzy* logics is the extension of any (weakly implicative) logic **L** by Baaz delta: the so-called *'Baaz companion'* of **L** is the weakest fuzzy logic that extends **L** and contains the Baaz delta connective (see [2]).

Similarly, one can define first-order variants of weakly implicative logics with the usual axioms of [9] for quantifiers (specification, its dual, and the quantifier shifts over implication). A stronger first-order variant can be defined for *fuzzy* logics with disjunction by adding the axiom (∀3) of [5]. General completeness results for large classes of such first-order *fuzzy* logics can be found in [1]. Uniform definitions and general properties of higher-order *fuzzy* logics, *fuzzy* logics with the identity predicate, and the notion of crispness in all *fuzzy* logics are currently being investigated by the authors. Both the results of [1,2] and preliminary results in the latter areas under investigation hint that the formal notion of *fuzzy* logic is fruitful and worth studying.

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